

The category of equiological spaces and the effective topos as homotopical quotients

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Abstract

We show that the two models of an extensional version of Martin-Löf type theory, those given by the category of equiological spaces and by the effective topos, are homotopical quotients of appropriate categories of 2-groupoids.

1 Introduction

The category of T_0 -spaces embeds fully in the category of equiological spaces; the category of equiological spaces is locally cartesian closed and the embedding functor preserves products and any exponential available in the original category. Thus the category of equiological spaces provides a nice extension of the category of T_0 -spaces. The effective topos is the categorical rendering of Kleene’s realizability model for intuitionistic logic, and is the first interesting example of a non-Grothendieck topos. We show that the category of equiological spaces is the homotopical quotient of a category of groupoids, and that the effective topos is the homotopical quotient of a category of 2-groupoids of partitioned assemblies.

Groupoids are a main tool in algebraic topology, see [Bro68] and groupoids were the first nontrivial models of the intensional version of Martin-Löf Type Theory in [HS98]. Moreover in recent years the Univalent Foundations Program, see [Uni13], has advocated a strong connection between algebraic topology and type theory.

Since both the category of equiological spaces and the effective topos are models of an extensional version of Martin-Löf type theory, it is useful to find that each comes from the “extensionalization” of a model of intensional type theory and that such a process is actually a homotopical quotient. We should stop here to point out that the meaning we adopt for an homotopical quotient of a category is in line with a suggestion in [CV98] and is the more naive notion obtained from an interval-like object than that derived from a Quillen model category—the main reason is that one of the two example categories we study

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is neither complete nor cocomplete. So, as a homotopical quotient, we shall consider a category obtained as a quotient category from a category \mathcal{C} with finite limits, as follows:

- there is a fixed **interval-like** object I , i.e. it has two global points $0: T \rightarrow I$ and $1: T \rightarrow I$ whose pushout

$$\begin{array}{ccc} T & \xrightarrow{1} & I \\ 0 \downarrow & & \downarrow 0' \\ I & \xrightarrow{1'} & I +_T I \end{array}$$

exists in \mathcal{C} and is stable under products, an arrow $\gamma: I \rightarrow I +_T I$ and an arrow $\iota: I \rightarrow I$ such that the four arrows together with the unique arrow $!: I \rightarrow T$ form an **equivalence co-span** in \mathcal{C} , i.e. the following diagrams commute

$$\begin{array}{ccc} T & \xrightarrow{0} & I \xleftarrow{1} T \\ & \searrow 1 & \downarrow \iota \swarrow 0 \\ & & I \end{array} \quad \begin{array}{ccc} T & \xrightarrow{0} & I \xleftarrow{1} T \\ & \searrow 0 & \downarrow \gamma \swarrow 1 \\ & & I \\ & \searrow 0' & \downarrow \gamma \swarrow 1' \\ & & I +_T I \end{array}$$

—note that there is also a necessarily commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{0} & I \xleftarrow{1} T \\ & \searrow \text{id}_T & \downarrow ! \swarrow \text{id}_T \\ & & T \end{array}$$

since T is terminal—;

- two arrows $f, g: X \rightarrow Y$ are identified in the quotient if there is an arrow $h: X \times I \rightarrow Y$ such that the following diagram commute

$$\begin{array}{ccccc} & X & & & \\ & \downarrow \langle \text{id}_X, 0 \rangle & \searrow f & & \\ X \times I & \xrightarrow{h} & Y & & \\ & \uparrow \langle \text{id}_X, 1 \rangle & \swarrow g & & \\ & X & & & \end{array}$$

The condition of the structure on I ensures that the identification is an equivalence relation on parallel arrows in \mathcal{C} .

It seems plausible that the categories we analyse in the following sustain suitable notions of fibrations, cofibrations and weak equivalences—in particular, that a map of the kind $\langle \text{id}_X, i \rangle: X \longrightarrow X \times I$, $i = 0, 1$, is a weak equivalence. But the categories are certainly not complete, nor cocomplete, and that prevents a direct comparison with standard homotopical quotients. It will be considered in future work.

We introduce the category of equilogical spaces in section 2 and we recall one of the presentations of the effective topos in section 3, reviewing properties which are needed in the following sections. In section 4 we determine a category \mathcal{A} of topological groupoids and an interval-like topological groupoid \mathbb{I} such that the homotopical quotient of \mathcal{A} determined by \mathbb{I} is equivalent to the category of equilogical spaces. In section 5 we produce a similar result for the effective topos using a category of 2-groupoids on partitioned assemblies.

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2 Equilogical spaces

Recall from [Sco96, BBS04] that an *equilogical space* $\mathcal{E} = (S_{\mathcal{E}}, \tau_{\mathcal{E}}, \sim_{\mathcal{E}})$ consists of a T_0 -space $(S_{\mathcal{E}}, \tau_{\mathcal{E}})$ and an equivalence relation $\sim_{\mathcal{E}} \subseteq S_{\mathcal{E}} \times S_{\mathcal{E}}$ on the points of the space.

A *map* $[f]: \mathcal{E} \longrightarrow \mathcal{F}$ *of equilogical spaces* is an equivalence class of continuous functions $f: (S_{\mathcal{E}}, \tau_{\mathcal{E}}) \longrightarrow (S_{\mathcal{F}}, \tau_{\mathcal{F}})$ preserving the equivalence relations, *i.e.* if $x \sim_{\mathcal{E}} x'$, then $f(x) \sim_{\mathcal{F}} f(x')$ for all x and x' in $S_{\mathcal{E}}$. For two such continuous functions $f, g: (S_{\mathcal{E}}, \tau_{\mathcal{E}}) \longrightarrow (S_{\mathcal{F}}, \tau_{\mathcal{F}})$, one sets f *equivalent to* g when $f(x) \sim_{\mathcal{F}} g(x)$ for all $x \in S_{\mathcal{E}}$.

Composition of maps of equilogical spaces $[f]: \mathcal{E} \longrightarrow \mathcal{F}$ and $[g]: \mathcal{F} \longrightarrow \mathcal{G}$ is given on (any of) their continuous representatives: $[g] \circ [f] := [g \circ f]$.

The data above determine a category *Equ* of equilogical spaces. There is a full embedding

$$Y: \text{Top}_0 \xrightarrow[\text{full}]{} \text{Equ}$$

which maps a T_0 -space (S, τ) to the equilogical space on (S, τ) with the diagonal relation, *i.e.* the equilogical space $(S, \tau, =_S)$.

The category *Equ* is a locally cartesian closed full extension of the category Top_0 of T_0 -spaces. In fact, it is the intersection of two other locally cartesian

closed full extensions of

$$\begin{array}{ccccc}
\mathcal{Top}_0 & \xleftarrow{\perp} & \mathcal{Equ}_C & \xleftarrow{\perp} & (\mathcal{Top}_0)_{\text{ex}} \\
\uparrow \downarrow \dashv & & \uparrow \downarrow \dashv & & \uparrow \downarrow \dashv \\
\mathcal{Top}_C & \xleftarrow{\perp} & \mathcal{Top}_{\text{reg}} & \xleftarrow{\perp} & \mathcal{Top}_{\text{ex}}
\end{array}$$

The exact completions $(\mathcal{Top}_0)_{\text{ex}}$ and $\mathcal{Top}_{\text{ex}}$ are pretoposes, while the regular completion $\mathcal{Top}_{\text{reg}}$ is a quasitopos, see [Ros00].

The product of equilogical spaces $\mathcal{E} \times \mathcal{F}$ is computed as expected taking the topological product $(S_{\mathcal{E}}, \tau_{\mathcal{E}}) \times (S_{\mathcal{F}}, \tau_{\mathcal{F}})$ and the equivalence relation

$$\langle a, b \rangle \sim_{\mathcal{E} \times \mathcal{F}} \langle a', b' \rangle \text{ when } a \sim_{\mathcal{E}} a' \text{ and } b \sim_{\mathcal{F}} b'.$$

The projections to the factors are obvious.

The construction of the exponential $\mathcal{F}^{\mathcal{E}}$ is less direct and we refer the reader to the basic sources [Sco76, Sco96, BBS04] as well as [BR14, BCRS98].

It is useful for the purpose of this paper to point out the strong similarity between the presentation of \mathcal{Equ} and that of $(\mathcal{Top}_0)_{\text{ex}}$. So recall from [CC82, Car95, FS91, CV98] that the exact completion \mathcal{C}_{ex} of a category \mathcal{C} with finite

limits is a quotient category of the full subcategory $\text{ES}(\mathcal{C})$ of the category $\mathcal{C}^{\bullet \rightarrow \bullet}$ of graphs in \mathcal{C} on the equivalence spans.

Recall that a (directed) graph in \mathcal{C} is a parallel pair $A_1 \xrightleftharpoons[d_2]{d_1} A_0$ of arrows of

\mathcal{C} and a homomorphism from the graph $A_1 \xrightleftharpoons[d_2]{d_1} A_0$ to the graph $B_1 \xrightleftharpoons[e_2]{e_1} B_0$

is a pair $(f_1: A_1 \rightarrow B_1, f_0: A_0 \rightarrow B_0)$ of arrows in \mathcal{C} such that the following diagram commutes

$$\begin{array}{ccccc}
A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_2} & A_0 \\
f_0 \downarrow & & f_1 \downarrow & & \downarrow f_0 \\
B_0 & \xleftarrow{e_1} & B_1 & \xrightarrow{e_2} & B_0.
\end{array}$$

An *equivalence span* is a graph $A_1 \xrightleftharpoons[d_2]{d_1} A_0$ in \mathcal{C} which is reflexive, symmetric, and endowed with a compatible operation on pairs of consecutive arcs, i.e. there are arrows $r: A_0 \rightarrow A_1$, $s: A_1 \rightarrow A_1$, and $t: A_1 \times_{A_0} A_1 \rightarrow A_1$,

where

$$\begin{array}{ccc} A_1 \times_{A_0} A_1 & \xrightarrow{d'_2} & A_1 \\ d'_1 \downarrow & & \downarrow d_1 \\ A_1 & \xrightarrow{d_2} & A_0 \end{array}$$

is a pullback in \mathcal{C} , such that the following diagrams commute:

$$\begin{array}{ccc} & A_0 & \\ \text{id}_{A_0} \swarrow & \downarrow r & \searrow \text{id}_{A_0} \\ A_0 & \xleftarrow{d_1} A_1 \xrightarrow{d_2} & A_0 \end{array} \quad \begin{array}{ccc} & A_1 & \\ d_2 \swarrow & \downarrow s & \searrow d_1 \\ A_0 & \xleftarrow{d_1} A_1 \xrightarrow{d_2} & A_0 \end{array}$$

$$\begin{array}{ccccc} & A_1 \times_{A_0} A_1 & & A_1 & \\ & \downarrow d'_1 & \downarrow t & \downarrow d'_2 & \\ & A_1 & & A_1 & \\ d_1 \swarrow & & & & \searrow d_2 \\ A_0 & \xleftarrow{d_1} A_1 \xrightarrow{d_2} & A_0. \end{array}$$

The quotient category \mathcal{C}_{ex} is obtained by identifying homomorphisms (f_1, f_0) and (g_1, g_0) from $A_1 \xrightarrow[d_2]{d_1} A_0$ to $B_1 \xrightarrow[e_2]{e_1} B_0$ if there is an arrow $h: A_0 \rightarrow B_1$ such that

$$\begin{array}{ccccc} & A_0 & & & \\ f_0 \swarrow & \downarrow h & \searrow g_0 & & \\ B_0 & \xleftarrow{e_1} B_1 \xrightarrow{e_2} & B_0 \end{array}$$

—nothing is asked of the other component.

The following proposition makes the similarity explicit.

2.1 Proposition. *The category \mathcal{Equ} is equivalent to the full subcategory \mathcal{A} of $(\mathcal{Top}_0)_{\text{ex}}$ on those equivalence spans $A_1 \xrightarrow[d_2]{d_1} A_0$ of topological spaces and continuous maps such that the pair $\langle d_1, d_2 \rangle: A_1 \rightarrow A_0 \times A_0$ is a subspace inclusion.*

Proof. Consider an equivalence span $A = A_1 \xrightarrow[d_2]{d_1} A_0$ of topological spaces and continuous maps such that the pair $\langle d_1, d_2 \rangle: A_1 \rightarrow A_0 \times A_0$ is a subspace inclusion. Note that the functions r , s and t requested by the definition of

equivalence span are unique, and determine that the subset $|A_1|$ of pairs of points of $|A_0|$ is an equivalence relation. Write $F(A)$ for the equilogical space which consists of the topological space A_0 and the equivalence relation $|A_1|$.

For a homomorphism (f_1, f_0) between two such equivalence spans, the component f_1 is uniquely determined by the other data as the restriction of the pair $\langle f_0, f_0 \rangle$, and ensures that f_0 is a representative of a map of equilogical spaces. Moreover, in the quotient category $(\mathbf{Top}_0)_{\text{ex}}$, the homomorphism (f_1, f_0) is identified with (g_1, g_0) precisely when $\langle f(x), g(x) \rangle$ is in A_1 for all points x in A_0 .

Thus the assignment $F([f_1, f_0]) = [f_0]$ is well defined, and determines a functor from \mathcal{A} to \mathbf{Equ} which is full and faithful.

To see that F is also bijective on objects, suppose $\mathcal{E} = (S_{\mathcal{E}}, \tau_{\mathcal{E}}, \sim_{\mathcal{E}})$ is an equilogical space. Consider the subspace topology $\sigma_{\mathcal{E}}$ on $\sim_{\mathcal{E}} \subseteq S_{\mathcal{E}} \times S_{\mathcal{E}}$ and the graph of topological spaces

$$(\sim_{\mathcal{E}}, \sigma_{\mathcal{E}}) \xrightleftharpoons[\pi_2]{\pi_1} (S_{\mathcal{E}}, \tau_{\mathcal{E}})$$

induced by the two projections. It is easy to check that it is an equivalence span and, by construction, the pair $\langle \pi_1, \pi_2 \rangle: (\sim_{\mathcal{E}}, \sigma_{\mathcal{E}}) \rightarrow (S_{\mathcal{E}}, \tau_{\mathcal{E}}) \times (S_{\mathcal{E}}, \tau_{\mathcal{E}})$ is a subspace inclusion. It is obvious that the functor F takes that equivalence span of \mathcal{A} to the equilogical space \mathcal{E} . \square

In the following, we shall refer to an equivalence span $A_1 \xrightleftharpoons[d_2]{d_1} A_0$ of topological spaces and continuous maps such that the pair $\langle d_1, d_2 \rangle: A_1 \rightarrow A_0$ is a subspace inclusion as a **spatial** equivalence span.

3 The effective topos

The effective topos $\mathcal{E}ff$ was introduced in [HJP80, Hyl82]. It was shown in [RR90] that $\mathcal{E}ff$ is (equivalent to) the exact completion of the category $\mathcal{P}A\mathbf{sm}$ of partitioned assemblies, see [CFS88].

A **partitioned assembly** is a function $\xi: X \rightarrow \mathbb{N}$; a **map** $\begin{array}{ccc} X & & Y \\ \downarrow \xi & \xrightarrow{f} & \downarrow \zeta \\ \mathbb{N} & & \mathbb{N} \end{array}$

of partitioned assemblies is a function $f: X \rightarrow Y$ such that there is a partial recursive function $\phi: \mathbb{N} \rightarrow \mathbb{N}$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \xi \downarrow & & \downarrow \zeta \\ \mathbb{N} & \xrightarrow{\phi} & \mathbb{N} \end{array}$$

In order to make sure that the exact completion introduced in section 2 can be applied to the category $\mathcal{P}A\mathbf{sm}$ we recall how finite limits can be obtained in that category.

The product of two partitioned assemblies is obtained by adopting some particular recursive encoding $\langle\langle n, m \rangle\rangle$ of pairs of numbers; the product partitioned

assembly of $\xi \downarrow_{\mathbb{N}}^X$ and $\zeta \downarrow_{\mathbb{N}}^Y$ is the function

$$(x, y) \mapsto \langle\langle \xi(x), \zeta(y) \rangle\rangle: X \times Y \longrightarrow \mathbb{N}$$

with obvious projections.

The equalizer of $\begin{array}{ccc} X & & Y \\ \downarrow \xi & \xRightarrow[f]{g} & \downarrow \zeta \\ \mathbb{N} & & \mathbb{N} \end{array}$ is the partitioned assembly $\xi \downarrow_E: E \longrightarrow \mathbb{N}$

where $E := \{x \in \mathbb{N} \mid f(x) = g(x)\}$ with the obvious inclusion into $\xi \downarrow_{\mathbb{N}}^X$.

The next result will be useful in the following.

3.1 Lemma. *Every equivalence span*

$$\begin{array}{ccc} A_1 & & A_0 \\ \downarrow \alpha_1 & \xRightarrow[d_2]{d_1} & \alpha_0 \downarrow \\ \mathbb{N} & & \mathbb{N} \end{array}$$

in $\mathcal{P}Asm_{\text{ex}}$ is isomorphic to one of the form

$$\begin{array}{ccc} E & & A_0 \\ \downarrow \epsilon & \xRightarrow[e_2]{e_1} & \alpha_0 \downarrow \\ \mathbb{N} & & \mathbb{N} \end{array}$$

such that the triple $\langle e_1, e_2, \epsilon \rangle$ is monic.

Proof. Consider an arbitrary equivalence span

$$\begin{array}{ccc} A_1 & & A_0 \\ \downarrow \alpha_1 & \xRightarrow[d_2]{d_1} & \alpha_0 \downarrow \\ \mathbb{N} & & \mathbb{N} \end{array}$$

in $\mathcal{P}Asm_{\text{ex}}$. So there are two partial recursive functions ϕ_1 and ϕ_2 such that the following diagram commutes

$$\begin{array}{ccccc} A_0 & \xleftarrow{d_1} & A_1 & \xrightarrow{d_2} & A_0 \\ \alpha_0 \downarrow & & \alpha_1 \downarrow & & \downarrow \alpha_0 \\ \mathbb{N} & \xleftarrow{\phi_1} & \mathbb{N} & \xrightarrow{\phi_2} & \mathbb{N}. \end{array}$$

Take E to be the image of the function $\langle d_1, d_2, \alpha_1 \rangle: A_1 \longrightarrow A_0 \times A_0 \times \mathbb{N}$, let $f: A_1 \longrightarrow E$ be the factoring surjection, and let $\epsilon := \pi_3 \upharpoonright_E: E \longrightarrow \mathbb{N}$. Let

$e_1, e_2: \downarrow_{\mathbb{N}}^E \epsilon \longrightarrow \alpha_0 \downarrow_{\mathbb{N}}^{A_0}$ be the first and second projection respectively. It is easy to see that it is an equivalence span.

Clearly f gives rise to a map of partitioned assemblies $\downarrow_{\mathbb{N}}^{A_1} \alpha_1 \xrightarrow{f} \epsilon \downarrow_{\mathbb{N}}^E$ since there is a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & E \\ \alpha_1 \downarrow & & \downarrow \epsilon \\ \mathbb{N} & \xrightarrow{\text{id}_{\mathbb{N}}} & \mathbb{N}. \end{array}$$

Moreover any section $s: E \rightarrow A_1$ of f (as a function of sets) is a map of partitioned assemblies $\downarrow_{\mathbb{N}}^E \epsilon \xrightarrow{s} \alpha_1 \downarrow_{\mathbb{N}}^{A_1}$ and a section of $\downarrow_{\mathbb{N}}^{A_1} \alpha_1 \xrightarrow{f} \epsilon \downarrow_{\mathbb{N}}^E$ in $\mathcal{P}\mathcal{A}sm$.

Thus an appeal to the axiom of choice yields the conclusion. \square

3.2 Remark. Note that the axiom of choice was used in a crucial way in 3.1 to determine an equivalence span of the required form and the requested isomorphism, but the proof that

$$\downarrow_{\mathbb{N}}^E \epsilon \xrightleftharpoons[e_2]{e_1} \alpha_0 \downarrow_{\mathbb{N}}^{A_0}$$

is an equivalence span does not require the use of the axiom of choice.

We conclude this brief review of the effective topos recalling a diagram of functors considered by Aurelio Carboni in [Car95] which shows how similar the situation is to that of topological spaces. Write $\mathcal{P}\mathcal{A}sm_0$ for the full subcategory of $\mathcal{P}\mathcal{A}sm$ on those partitioned assemblies which are 1-1 (functions). This is clearly equivalent to the category $\mathcal{P}\mathcal{R}$ whose objects are subsets of \mathbb{N} and whose arrows are restriction of partial recursive functions between those, total on the domain.

$$\begin{array}{ccccc} \mathcal{P}\mathcal{A}sm_0 & \hookrightarrow & \mathcal{P}\mathcal{E}\mathcal{R}_{\mathcal{C}} & \xleftarrow{\perp} & (\mathcal{P}\mathcal{A}sm_0)_{\text{ex}} \\ \uparrow \downarrow \dashv & & \uparrow \downarrow \dashv & & \uparrow \downarrow \dashv \\ \mathcal{P}\mathcal{A}sm & \hookrightarrow & \mathcal{P}\mathcal{A}sm_{\text{reg}} & \xleftarrow{\perp} & \mathcal{P}\mathcal{A}sm_{\text{ex}}. \end{array}$$

In the diagram of full subcategories of $\mathcal{E}ff$, the exact completion $\mathcal{P}\mathcal{A}sm_{\text{ex}}$ is itself the effective topos; $\mathcal{P}\mathcal{A}sm_{\text{reg}}$ is the full subcategory of $\mathcal{E}ff$ on the $\neg\neg$ -separated objects; $(\mathcal{P}\mathcal{A}sm_0)_{\text{ex}}$ is the full subcategory of $\mathcal{E}ff$ on the discrete objects—i.e. subquotients of the natural number object of $\mathcal{E}ff$, see [HRR90]—;

and \mathcal{PER} is the intersection of the last two, the full subcategory of \mathcal{Eff} on the $\neg\neg$ -separated subquotients of the natural number object of \mathcal{Eff} , also known as “partial equivalence relations on \mathbb{N} ”, see [Hyl88]. As is shown in [Car95], this last is not the regular completion of $\mathcal{PR} \equiv \mathcal{PAsm}_0$. A similar remark applies to \mathcal{Equ} and $(\mathcal{Top}_0)_{\text{reg}}$ which are not equivalent—this corrects a hastily mistaken, happily irrelevant statement in [BR14].

4 Groupoids

Consider a category \mathcal{C} with pullbacks. A **groupoid** \mathbb{G} in \mathcal{C} is a graph $G_1 \rightrightarrows^{d_1}_{d_2} G_0$ of objects and arrows in \mathcal{C} together with three more arrows

$$i: G_0 \longrightarrow G_1 \quad c: G_1 \times_{G_0} G_1 \longrightarrow G_1 \quad s: G_1 \longrightarrow G_1$$

where

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{d'_2} & G_1 \\ d'_1 \downarrow & & \downarrow d_1 \\ G_1 & \xrightarrow{d_2} & G_0 \end{array}$$

is a pullback in \mathcal{C} , such that

- the graph $G_1 \rightrightarrows^{d_1}_{d_2} G_0$ with i and c is a category object in \mathcal{C} ,
- s is an involution which makes every arrow an isomorphism.

The notions of **functor** of groupoids in \mathcal{C} is obvious as well as that of **natural transformation**. It is straightforward to check that a functor between groupoids preserves the involution which makes every arrow an isomorphism.

We have already available a large number of examples as follows from the next property.

4.1 Proposition. Let $G_1 \rightrightarrows^{d_1}_{d_2} G_0$ be a graph in \mathcal{C} with arrows $r: G_0 \longrightarrow G_1$,

$t: G_1 \times_{G_0} G_1 \longrightarrow G_1$, and $s: G_1 \longrightarrow G_1$ such that the diagrams

$$\begin{array}{ccc}
 & G_0 & \\
 \text{id}_{G_0} \swarrow & \downarrow r & \searrow \text{id}_{G_0} \\
 G_0 & \xleftarrow{d_1} G_1 \xrightarrow{d_2} & G_0
 \end{array}
 \quad
 \begin{array}{ccc}
 & G_1 & \\
 d_2 \swarrow & \downarrow s & \searrow d_1 \\
 G_0 & \xleftarrow{d_1} G_1 \xrightarrow{d_2} & G_0
 \end{array}$$

$$\begin{array}{ccc}
 & G_1 \times_{G_0} G_1 & \\
 d'_1 \swarrow & \downarrow t & \searrow d'_2 \\
 G_1 & & G_1 \\
 d_1 \swarrow & & \searrow d_2 \\
 G_0 & \xleftarrow{d_1} G_1 \xrightarrow{d_2} & G_0
 \end{array}$$

commute. If the pair $G_1 \xrightleftharpoons[d_2]{d_1} G_0$ is jointly monic, then

(i) the structure given by

$$\begin{array}{ccc}
 & & r \\
 & & \downarrow \\
 G_2 & \xrightarrow{t} & G_1 \xrightleftharpoons[d_2]{d_1} G_0 \\
 & \uparrow s & \\
 & &
 \end{array}$$

is a groupoid \mathbb{G} in \mathcal{C} ,

(ii) for any groupoid \mathbb{H} in \mathcal{C} , a graph-homomorphism from the underlying

graph $H_1 \xrightleftharpoons[e_2]{e_1} H_0$ of \mathbb{H} to $G_1 \xrightleftharpoons[d_2]{d_1} G_0$ is also a functor from \mathbb{H} to \mathbb{G} ,

(iii) for any groupoid \mathbb{H} in \mathcal{C} , let (f_1, f_0) and (g_1, g_0) be functors from the groupoid \mathbb{H} to the groupoid \mathbb{G} . Then an arrow $a: H_0 \longrightarrow G_1$ such that

$$\begin{array}{ccc}
 & H_0 & \\
 f_0 \swarrow & \downarrow a & \searrow g_0 \\
 G_0 & \xleftarrow{d_1} G_1 \xrightarrow{d_2} & G_0
 \end{array}$$

is a natural transformation from (f_1, f_0) to (g_1, g_0) .

Proof. Straightforward. \square

4.2 Corollary. Every subspatial equivalence span is a groupoid in \mathcal{Top}_0 . Every representative of an arrow in \mathcal{A} is a functor between the groupoids.

Consider the interval-like groupoid

$$\mathbb{I} := \circ \rightrightarrows \circ$$

with the discrete topology. A natural transformation as in 4.1(iii) is the same as a functor $\mathbb{H} \times \mathbb{I} \longrightarrow \mathbb{G}$. Thanks to 2.1, we may rephrase Corollary 4.2 as follows.

4.3 Theorem. *The category $\mathcal{E}qu$ of equilogical spaces is the homotopical quotient of the category \mathcal{A} of topological groupoids.*

5 2-groupoids

A similar case can be made for the effective topos. We prove in the following that it is the homotopical quotient of a category of higher groupoids in $\mathcal{P}asm$.

Consider a category \mathcal{C} with pullbacks. A **2-groupoid** \mathbb{G} in \mathcal{C} is a 2-graph

$$\begin{array}{ccc} G_2 & \begin{array}{c} \xrightarrow{d_{21}} \\ \xrightarrow{d_{22}} \end{array} & G_1 \\ & \searrow \begin{array}{c} d_{11}d_{21} = d_{11}d_{22} \\ d_{12}d_{21} = d_{12}d_{22} \end{array} & \downarrow \begin{array}{c} d_{12} \\ d_{11} \end{array} \\ & & G_0 \end{array}$$

of objects and arrows in \mathcal{C} together with arrows

$$\begin{array}{lll} i_1: G_0 \longrightarrow G_1 & c_1: G_1 \times_{G_0} G_1 \longrightarrow G_1 & s_1: G_1 \longrightarrow G_1 \\ i_2: G_1 \longrightarrow G_2 & c_2: G_2 \times_{G_1} G_2 \longrightarrow G_2 & s_2: G_2 \longrightarrow G_2 \\ & c'_2: G_2 \times_{G_0} G_2 \longrightarrow G_2 & q: G_1 \longrightarrow G_2 \end{array}$$

where

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 \longrightarrow G_1 & G_2 \times_{G_1} G_2 \longrightarrow G_2 & G_2 \times_{G_0} G_2 \longrightarrow G_2 \\ \downarrow & \downarrow & \downarrow \\ G_1 \xrightarrow{d_{12}} G_0 & G_2 \xrightarrow{d_{22}} G_1 & G_2 \xrightarrow{d_{12}d_{22}} G_0 \end{array}$$

are pullbacks in \mathcal{C} , such that

- the 2-graph $G_2 \begin{smallmatrix} \xrightarrow{d_{21}} \\ \xrightarrow{d_{22}} \end{smallmatrix} G_1 \begin{smallmatrix} \xrightarrow{d_{11}} \\ \xrightarrow{d_{12}} \end{smallmatrix} G_0$ with i_1, c_1, i_2, c_2, c'_2 is a 2-category object in \mathcal{C} ,
- s_1 is an involution which makes every 1-arrow an equivalence via the pair of arrows given by q ,

- s_2 is an involution which makes every 2-arrow an iso.

The notions of **2-functor** of 2-groupoids in \mathcal{C} is obvious as well as that of **2-transformation**.

Consider the 2-category $\text{Grpd}(\mathcal{P}\mathcal{A}sm)$ of 2-groupoids in $\mathcal{P}\mathcal{A}sm$ with 2-functors and 2-transformations. Clearly, the underlying graph of a 2-groupoid \mathbb{G} of $\mathcal{P}\mathcal{A}sm$ is an equivalence span in $\mathcal{P}\mathcal{A}sm$, thus an object of $\mathcal{E}ff$. This extends directly to a functor $U: \text{Grpd}(\mathcal{P}\mathcal{A}sm) \rightarrow \mathcal{E}ff$.

5.1 Theorem. *The functor $U: \text{Grpd}(\mathcal{P}\mathcal{A}sm) \rightarrow \mathcal{E}ff$ is essentially surjective.*

Proof. Consider an object in $\mathcal{E}ff$, by 3.1 we can assume without loss of gen-

erality that it is an equivalence span $\begin{array}{ccc} A_1 & \xrightarrow[a_2]{a_1} & A_0 \\ \downarrow \alpha_1 & & \downarrow \alpha_0 \\ \mathbb{N} & & \mathbb{N} \end{array}$ in $\mathcal{P}\mathcal{A}sm$ such that

the triple $\langle a_1, a_2, \alpha \rangle$ is monic. Take the free dagger category on that graph in $\mathcal{P}\mathcal{A}sm$ —by a **dagger category** we mean a category together with a involutive contravariant functor which is the identity on objects. It consists of $\begin{array}{c} A_0 \\ \downarrow \alpha_0 \\ \mathbb{N} \end{array}$ as

objects of objects. The object of 1-arrows is $\alpha^\wedge \downarrow \mathbb{N}$ where A^\wedge consists of the

zigzag paths in the graph $A_1 \xrightarrow[a_2]{a_1} A_0$. By a zigzag path in the graph we mean a list which is either of the form $\langle x \rangle$ where $x \in A_0$ or

$$\langle x_0, e_1, i_1, x_1, e_2, i_2, x_2, \dots, x_n, e_{n+1}, i_{n+1}, x_{n+1} \rangle,$$

where

- $x_\ell \in A_0$ for $0 \leq \ell \leq n+1$,
- $e_\ell \in A_1$ for $1 \leq \ell \leq n+1$,
- $i_\ell \in \{0, 1\}$ for $1 \leq \ell \leq n+1$,
- for $0 \leq \ell \leq n$, if $i_\ell = 0$, then $\langle x_\ell, x_{\ell+1}, e_{\ell+1} \rangle \in A_1$,
- for $0 \leq \ell \leq n$, if $i_\ell = 1$, then $\langle x_{\ell+1}, x_\ell, e_{\ell+1} \rangle \in A_1$.

Intuitively, if one considers a triple $\langle x, x', e \rangle \in A_1$ as an edge e from the source

x to the target x' in the graph $A_1 \xrightarrow[a_2]{a_1} A_0$, then the zigzag

$$\langle x_0, e_1, i_1, x_1, e_2, i_2, x_2, \dots, x_n, e_{n+1}, i_{n+1}, x_{n+1} \rangle$$

is a mixed-directional path of edges from the vertex x_0 to the vertex x_{n+1} where each edge e_ℓ between x_ℓ and $x_{\ell+1}$ is marked with either 0 or 1: if the mark is 0, e_ℓ goes from x_ℓ to $x_{\ell+1}$ in the original graph; if the mark is 1, e_ℓ goes from

$x_{\ell+1}$ to x_ℓ . The function α^\wedge is defined by mapping a zigzag to the encoding of the list of its numerical components:

$$\begin{aligned}\alpha^\wedge(\langle x \rangle) &:= \langle\langle 0, \alpha_0(x) \rangle\rangle \\ \alpha^\wedge(\langle x_0, e_1, i_1, \dots, x_n, e_{n+1}, i_{n+1}, x_{n+1} \rangle) &:= \\ &:= \langle\langle n+1, \langle\alpha^\wedge(\langle x_0, e_1, i_1, \dots, x_n \rangle), \langle\langle e_{n+1}, i_{n+1} \rangle\rangle, a_0(x_{n+1}) \rangle\rangle\rangle.\end{aligned}$$

The structure of dagger category in $\mathcal{P}\mathcal{A}sm$ is obvious, changing each i_ℓ with $\sigma(i_\ell)$ where $\sigma: \{0, 1\} \rightarrow \{0, 1\}$ swaps 0 with 1. The object of 2-arrows $\alpha^- \downarrow_{\mathbb{N}}^{A^\mathfrak{M}}$ is formed by taking the total relation on each 1-homset, where $A^\mathfrak{M} := A^\wedge \times_{A_0} A^\wedge$. Explicitly, $A^\mathfrak{M}$ consists of all pairs of zigzags

$$\langle\langle x_0, e, \dots, x_n \rangle\rangle, \langle\langle x_0, e', \dots, x_n \rangle\rangle$$

between each two given vertices x and x' ; clearly all 2-diagrams commute as there is at most one 2-arrow from an 1-arrow to another. In this way, the dagger functor becomes the involution which makes every 1-arrow an equivalence. It is easy to see that that gives a 2-groupoid on the given span in $\mathcal{P}\mathcal{A}sm$ and that

$$\text{the functor } U \text{ takes it to a span which is isomorphic to } \begin{array}{ccc} A_1 & \xrightarrow{e_1} & A_0 \\ \downarrow \alpha_1 & \xRightarrow{\quad} & \alpha_0 \downarrow \\ \mathbb{N} & \xrightarrow{e_2} & \mathbb{N} \end{array} . \quad \square$$

We shall refer to a 2-groupoid like that produced in the proof of 5.1 as a **numeric** 2-groupoid as all edges are denoted by numbers. More precisely, it is a 2-groupoid \mathbb{G} in $\mathcal{P}\mathcal{A}sm$ such that its underlying category in $\mathcal{P}\mathcal{A}sm$

$$G_1 \xRightarrow[d_{12}]{d_{11}} G_0$$

is a free dagger category and \mathbb{G} embeds, fully at level 2, into the 2-groupoid

$$G_0 \times G_0 \times \mathbb{N} \times \mathbb{N} \xRightarrow[\pi_{124}]{\pi_{123}} G_0 \times G_0 \times \mathbb{N} \xRightarrow[\pi_2]{\pi_1} G_0$$

where π_{123} and π_{124} are the projections deleting the fourth and third component, respectively.

5.2 Theorem. *The functor $U: \text{Grpd}(\mathcal{P}\mathcal{A}sm) \rightarrow \mathcal{E}ff$ restricts to a homotopical quotient of the full subcategory \mathcal{N} on the numeric 2-groupoids.*

Proof. Suppose that \mathbb{G} and \mathbb{H} are numeric groupoids. Since \mathbb{G} is a free dagger category and all 2-diagrams commute in \mathbb{H} , it is easy to see that every arrow $[f]: U(\mathbb{G}) \rightarrow U(\mathbb{H})$ in $\mathcal{E}ff$ has a representative which is a 2-functor $F: \mathbb{G} \rightarrow \mathbb{H}$. To see that the functor $U: \text{Grpd}(\mathcal{P}\mathcal{A}sm) \rightarrow \mathcal{E}ff$ restricted to \mathcal{N} is indeed a homotopical quotient, consider the interval-like groupoid \mathbb{I} : it is the free dagger category on the graph in $\mathcal{P}\mathcal{A}sm$ on $T + T$ with two (disjoint) nodes and a single

edge u connecting one with the other, with all possible 2-arrows. It is clearly a numeric 2-groupoid. Consider now two functors $F, F': \mathbb{G} \rightarrow \mathbb{H}$ such that $U(F) = U(F')$; in other words, there is a map $k: G_0 \rightarrow H_1$ in $\mathcal{P}\mathcal{A}sm$ such that

$$F_0 = d_{11}^{\mathbb{H}} \circ k \quad \text{and} \quad F'_0 = d_{12}^{\mathbb{H}} \circ k.$$

Note that the 1-category underlying the 2-groupoid $\mathbb{G} \times \mathbb{I}$ is a retract of a free dagger category. Using k to act on the generating arrow of \mathbb{I} as follows

$$\begin{array}{ccc} \langle x, 0 \rangle & \xrightarrow{\quad} & F_0(x) \\ \downarrow \langle \langle x \rangle, u \rangle & \xrightarrow{\quad} & k(x) \downarrow \\ \langle x, 1 \rangle & \xrightarrow{\quad} & F'_0(x) \end{array}$$

by freeness it is easy to obtain a functor $K: \mathbb{G} \times \mathbb{I} \rightarrow \mathbb{H}$ which gives a homotopy from F to F' . \square

References

- [BBS04] A. Bauer, L. Birkedal, and D.S. Scott. Equiological spaces. *Theoret. Comput. Sci.*, 315(1):35–59, 2004.
- [BCRS98] L. Birkedal, A. Carboni, G. Rosolini, and D.S. Scott. Type theory via exact categories. In V. Pratt, editor, *Proc. 13th Symposium in Logic in Computer Science*, pages 188–198, Indianapolis, 1998. I.E.E.E. Computer Society.
- [BR14] A. Bucalo and G. Rosolini. Sobriety for equiological spaces. *Theoret. Comput. Sci.*, 546:93–98, 2014.
- [Bro68] R. Brown. *Elements of modern topology*. McGraw Hill, 1968.
- [Car95] A. Carboni. Some free constructions in realizability and proof theory. *J. Pure Appl. Algebra*, 103:117–148, 1995.
- [CC82] A. Carboni and R. Celia Magno. The free exact category on a left exact one. *J. Aust. Math. Soc.*, 33(A):295–301, 1982.
- [CFS88] A. Carboni, P.J. Freyd, and A. Scedrov. A categorical approach to realizability and polymorphic types. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Mathematical Foundations of Programming Language Semantics*, volume 298 of *Lectures Notes in Comput. Sci.*, pages 23–42, New Orleans, 1988. Springer-Verlag.
- [CV98] A. Carboni and E.M. Vitale. Regular and exact completions. *J. Pure Appl. Algebra*, 125:79–117, 1998.

- [FS91] P.J. Freyd and A. Scedrov. *Categories Allegories*. North Holland Publishing Company, 1991.
- [HJP80] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos Theory. *Math. Proc. Camb. Phil. Soc.*, 88:205–232, 1980.
- [HRR90] J.M.E. Hyland, E.P. Robinson, and G. Rosolini. The discrete objects in the effective topos. *Proc. Lond. Math. Soc.*, 60:1–36, 1990.
- [HS98] M. Hofmann and Th. Streicher. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory (Venice, 1995)*, volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, 1998.
- [Hyl82] J.M.E. Hyland. The effective topos. In A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, pages 165–216. North Holland Publishing Company, 1982.
- [Hyl88] J.M.E. Hyland. A small complete category. *Ann. Pure Appl. Logic*, 40:135–165, 1988.
- [Ros00] G. Rosolini. Equilogical spaces and filter spaces. *Rend. Circ. Mat. Palermo*, 64(suppl.):157–175, 2000.
- [RR90] E.P. Robinson and G. Rosolini. Colimit completions and the effective topos. *J. Symb. Logic*, 55:678–699, 1990.
- [Sco76] D.S. Scott. Data types as lattices. *SIAM J. Comput.*, 5(3):522–587, 1976.
- [Sco96] D.S. Scott. A new category? Domains, spaces and equivalence relations. Manuscript, 1996.
- [Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013.